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2008 J. Phys. A: Math. Theor. 41 145304

(http://iopscience.iop.org/1751-8121/41/14/145304)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 145304 (8pp) doi:10.1088/1751-8113/41/14/145304

Stabilizer quantum codes over the Clifford algebra

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Received 10 August 2007, in final form 24 November 2007
Published 26 March 2008
Online at stacks.iop.org/JPhysA/41/145304

Abstract
The key problem for constructing a stabilizer quantum code is how to create a set of generators for the stabilizer of the stabilizer quantum code, i.e. check matrix. In this paper, we suggest an approach based on the Clifford algebra to create the check matrix for the stabilizer quantum codes. In the proposed approach, the recursive relation of the matrix transform over the Clifford algebra is employed to generate the check matrix. With the proposed approach, a quantum code with any length can be constructed easily. Especially some new codes, which are impossible via previous approaches, are constructed.

PACS numbers: 03.67.Pp, 03.67.Hk, 03.65.−w

1. Introduction

Theory of quantum error correction codes (QECC) denoted by $[[N, k, d]]$ demonstrates a formal possibility of efficiently storing and manipulating data for arbitrarily long time even in the presence of noise that is below a certain threshold. Since the initial discovery and the general descriptions of QECC were presented [1, 2], researchers have made great progress on analyzing physical principles [3–8] along with constructing various quantum codes [9–14]. As is well known, QECC have become one of the significant ingredients in quantum computation [15], quantum signal processing [16] and quantum communication [17].

Currently, almost all of the advanced code constructions may be categorized into two classes according to construction methods and principles, i.e. Calderbank–Shor–Steane's (CSS) structure [1, 2] and the stabilizer quantum code's structure [13]. A thorough discussion of the principles of quantum coding theory was presented in [18], together with a tabulation of codes and bounds on the minimum distance for codeword length N up to 30 qubits. However, the construction approaches are somewhat complex. In addition, for a larger N there is less progress, and only a few general code constructions are known. It has been shown that CSS codes may exist as N goes to infinity. But it is impossible to construct this kind of codes for large N with great efficiency and speed. To construct the stabilizer quantum codes, a set of generators for the stabilizer of the stabilizer quantum codes should be designed first. However, if N is too large, it is very difficult to obtain the generators required to construct the stabilizer quantum codes.

In this paper, we present a new approach for generating a set of generators of the stabilizer of the quantum codes over the Clifford algebra. In the proposed approach, the set of generators of the stabilizer of the quantum code may be easily created; subsequently, a quantum code with any length N can be constructed easily. Especially some new codes, which are impossible to obtain via previous approaches, are constructed. The resulting codes are more efficient with better parameters than the previous quantum codes.

2. Constructing quantum codes

According to the depolarizing channel, there are four basic operators acting on a single qubit, i.e. $I, X = \sigma_x, Z = \sigma_z$ and $Y = ZX$, where σ_x and σ_z are Pauli matrix components and I is the identity operator. In an N-qubit depolarizing channel, an arbitrary operator acting on N-qubits belongs to Pauli group:

$$\mathcal{P}_N = \left\{E_i^{\otimes N} : E_i \in \{I, X, Z, Y\}, 1 \leqslant i \leqslant N\right\}, \tag{1}$$

where $\otimes N$ denotes N-fold tensor product.

To write the tensor product of Pauli matrices acting on N-qubits, we introduce the notation $X_{\vec{a}} Z_{\vec{b}}$ to denote an operator $\mathcal{E} \in \mathcal{P}_N$, where $\vec{a}$ and $\vec{b}$ are N-bit binary vectors. For example,

$$\mathcal{E} = X \otimes I \otimes Z \otimes Y \otimes X \tag{2}$$

may be expressed as $(\vec{a}|\vec{b}) = (10011|00110)$. Therefore, any operator in $\mathcal{P}_N$ can be uniquely denoted by a concatenated $2N$-dimension vector:

$$(\vec{a}|\vec{b}) = (a_1, a_2, \ldots, a_N|b_1, b_2, \ldots, b_N). \tag{3}$$

Clifford algebra $Cl(2, \mathbb{C})$ is isomorphic with algebra $\mathbb{C}(2 \times 2)$ of all complex 2×2 matrices, where $\mathbb{C}$ is a complex space. From what is described above, a linear transformation of 2^N-dimension space can be represented by the Clifford algebra with $2N$ generators, namely any quantum operator in $\mathcal{P}_N$ can be expressed by an element of $Cl(2N, \mathbb{C})$. So, a Clifford algebra $Cl(2N, \mathbb{C})$ may be generated by $2N$ generators $G^{(1)}, \ldots, G^{(2N)}$ with property

$$G^{(i)}G^{(j)} + G^{(j)}G^{(i)} = 2\delta_{ij} I_N, \tag{4}$$

for $i \neq j$ and $1 \leqslant i, j \leqslant 2N$, where these $2N$ generators may correspond to N operators in $\mathcal{P}_N$. This motivates us to consider the construction of stabilizer codes over the Clifford algebra. In the following, we investigate how to construct the generator matrix of the stabilizer quantum code over the Clifford algebra.

The stabilizer quantum code with parameters $[[N, k, d]]$ can be constructed from the stabilizer denoted by

$$\mathcal{S} = \left\{\prod_{i=1}^{N-k} \left(I + \mathcal{G}_i^{m_i}\right) : m_i \in \{0, 1\}\right\}, \tag{5}$$

where $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{N-k}$ are $N-k$ commuting generators of the stabilizer. The constructed code can correct up to $t = (d-1)/2$ quantum errors. Code words of the quantum code are states corresponding to simultaneous eigenvectors associated with eigenvalue '+1' of all operators in $\mathcal{S}$, i.e.

$$\tilde{C}(\mathcal{S}) = \{|C\rangle : |C\rangle = \hat{S}|C\rangle, \hat{S} \in \mathcal{S}\}, \tag{6}$$

where $|C\rangle = |C_1 \cdots C_N\rangle$. Therefore, to construct a stabilizer quantum code, one should first design $N-k$ generators of $\mathcal{S}$ [3]. These generators of the stabilizer compose the following check matrix:

$$\mathcal{G}_{N-k} = (\mathcal{G}^x \quad \mathcal{G}^z)_{(N-k)\times 2N} = \begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \\ \vdots \\ \mathcal{G}_{N-k} \end{pmatrix}, \tag{7}$$

where

$$\mathcal{G}^x = \begin{pmatrix} g^x_{11} & \cdots & g^x_{1N} \\ g^x_{21} & \cdots & g^x_{2N} \\ \vdots & \cdots & \vdots \\ g^x_{(N-k)1} & \cdots & g^x_{(N-k)N} \end{pmatrix}_{(N-k)\times N} \tag{8}$$

and

$$\mathcal{G}^z = \begin{pmatrix} g^z_{11} & \cdots & g^z_{1N} \\ g^z_{21} & \cdots & g^z_{2N} \\ \vdots & \cdots & \vdots \\ g^z_{(N-k)1} & \cdots & g^z_{(N-k)N} \end{pmatrix}_{(N-k)\times N}. \tag{9}$$

Denote by $\mathcal{G}_{N-k}$ a check matrix of quantum codes $[[N, k, d]]$; any two generators of $\mathcal{G}_{N-k}$ commute exactly when

$$\sum_{\tau=1}^{N} g^x_{i\tau} g^z_{i\tau} = 0 \qquad \bmod 2, \tag{10}$$

$$\sum_{\tau=1}^{N} g^x_{i\tau} g^z_{j\tau} + \sum_{\tau=1}^{N} g^z_{i\tau} g^x_{j\tau} = 0 \qquad \bmod 2, \qquad i \neq j, \tag{11}$$

where g^x_{st} and g^z_{st} are over $\mathbb{Z}_2$, and $1 \leqslant s \leqslant N-k$, $1 \leqslant t \leqslant N$. Namely, the symplectic inner product of two elements is zero [3].

To construct a quantum code $[[N, k, d]]$, one should choose suitable $2N$ generators for the Clifford algebra $Cl(2N, \mathbb{C})$ and achieve an approach for constructing the check matrix of code.

Theorem 2.1. *Choose a suitable set of generators for the Clifford algebra so that condition (4) is satisfied. Then, a quantum code with parameters* $[[N, k, d]]$ *may be obtained.*

Proof . Choose $2N$ elements $G^{(1)}, \ldots, G^{(2N)}$ in $Cl(2N, \mathbb{C})$ to satisfy condition (4). We know that it is an isomorphism $Cl(2N, \mathbb{C}) \cong \mathbb{C}(2 \times 2)^{\otimes N}$. So, there are N operators $\mathcal{X}_1, \ldots, \mathcal{X}_N$ in $\mathcal{P}_N$ corresponding to these $2N$ elements. To encode k-bit message, one should first append the message onto k operators among $\mathcal{X}_1, \ldots, \mathcal{X}_N$ and then construct $N-k$ generators of stabilizer. Subsequently, the k-bit message may be encoded into N-qubits, i.e. one may obtain a quantum code with parameters $[[N, k, d]]$. □

Theorem 2.1 shows that a stabilizer quantum code can be constructed over the Clifford algebra. So, the key problem is how to choose a suitable generators for the Clifford algebra, and construct the check matrix (i.e. the generators of stabilizer). In the following, we present an approach to solve this problem using the Pauli matrices and matrix transform.

First, we consider an approach of obtaining the elements for the Clifford algebra, which is based on Pauli matrices. Let Pauli matrices $\sigma_0, \sigma_x, \sigma_y, \sigma_z$ be the Clifford algebra basis elements. Making use of these matrices, we may construct the following $2N$ generators with size $2^N \times 2^N$ for the Clifford algebra $Cl(2 \times 2)^{\otimes N}$:

$$\begin{aligned} G^{(1)} &= \sigma_0^{\otimes(N-1)} \otimes \sigma_x \\ &\vdots \\ G^{(2k)} &= \sigma_0^{\otimes(N-k-1)} \otimes \sigma_x \otimes (\sigma_z)^{\otimes(k-1)} \\ G^{(2k+1)} &= \sigma_0^{\otimes(N-k-1)} \otimes \sigma_y \otimes (\sigma_z)^{\otimes(k-1)} \\ &\vdots \\ G^{(2N)} &= (\sigma_z)^{\otimes(N)}, \end{aligned} \tag{12}$$

where $k = 0, 1, 2, \ldots, N-1$. One may easily check that the set $\{G^{(1)}, \ldots, G^{(2N)}\}$ satisfies condition (4). According to theorem 2.1, these representations of the Clifford algebra may hence be employed to generate the check matrix for the quantum codes.

Then we consider the approach of obtaining the generators of stabilizer, which is based on matrix transform. Generally, some well-known matrix transforms such as Hadamard transform [19], discrete Fourier transform (DFT) and Jacket transform [20] have a useful recursive relation:

$$J_{p^m} = \prod_{i=1}^{m} I_{p^{m-i}} \otimes J_p \otimes I_{p^{i-1}}, \tag{13}$$

where p is a positive integer and $p \geqslant 2$. Equation (13) shows that the high size matrices can be obtained using the lowest order matrix J_p. Thus if the lowest order matrices J_p are orthogonal, i.e. equation (4) is satisfied, then the high size matrices obtained are also orthogonal. Therefore, suitable Jacket matrices may be chosen for the Clifford algebra. This result may be expressed using the following theorem.

Theorem 2.2. *If N matrices J_p with size p satisfy $J_p^T J_p + J_p J_p^T = 0$, then*

$$J_K^T J_K + J_K J_K^T = 0, \tag{14}$$

where $K = p^N$.

Proof . Employing equation (13), we have

$$J_K^T = \prod_{i=1}^{N} I_{p^{N-i}} \otimes J_p^T \otimes I_{p^{i-1}}, \tag{15}$$

$$J_K^T J_K = \prod_{i=1, j=1}^{N} (I_{p^{N-i}} I_{p^{N-j}}) \otimes \left(J_p^T J_p\right) \otimes (I_{p^{i-1}} I_{p^{j-1}}) \tag{16}$$

and

$$J_K J_K^T = \prod_{i=1, j=1}^{N} (I_{p^{N-i}} I_{p^{N-j}}) \otimes \left(J_p J_p^T\right) \otimes \left(I_{p^{i-1}} I_{p^{j-1}}\right). \tag{17}$$

Thus,

$$J_K^T J_K + J_K J_K^T = \prod_{i=1, j=1}^{N} (I_{p^{N-i}} I_{p^{N-j}}) \otimes \left(J_p^T J_p + J_p J_p^T\right) \otimes \left(I_{p^{i-1}} I_{p^{j-1}}\right). \tag{18}$$

Obviously, if $J_p^T J_p + J_p J_p^T = 0$ then equation (14) exists. □

Theorem 2.2 shows that if the lowest order matrices satisfy the condition in equations (10) and (11), then the high size generators obtained with the recursive relations also follow this condition. This provides a very convenient way for the construction of stabilizer quantum codes, since small size matrices which satisfy the condition in equations (10) and (11) can be easily found. Subsequently, a larger length stabilizer quantum code may be constructed using the recursive relation of matrices over the Clifford algebra.

After generators of stabilizer of the quantum codes have been generated, the stabilizer quantum code $[[N, k]]$ can be constructed. Generally, the proposed approach may be described as follows. Over the Clifford algebra, one obtains the concatenated matrix $G_N = (G^x\ G^z)$. Subsequently, a check matrix $\mathcal{G}_{N-k}$ is achieved according to theorem 2.1. Then, a set of generators $\mathcal{M}_{\mathcal{S}} = \{\mathcal{G}_i : 1 \leqslant i \leqslant N-k\}$ for the stabilizer $\mathcal{S}$ in equation (5) are obtained. Therefore, the normalizer of $\mathcal{S}$ can be denoted by

$$\mathcal{N}(\mathcal{S}) = \{\mathcal{E}_j : \mathcal{E}_j\mathcal{G}_i = \mathcal{G}_i\mathcal{E}_j, \mathcal{E}_j \in \mathbf{P}_N, \mathcal{G}_i \in \mathcal{S}\}. \tag{19}$$

One may easily find that there are $N+k$ mutually commuting independent generators of $\mathcal{N}(\mathcal{S})$ denoted by $\mathcal{E}_1, \ldots, \mathcal{E}_{n+k}$. For $\mathcal{E}_i, \mathcal{E}_j \in \mathcal{N}(\mathcal{S})$, it is easy to prove that both $\mathcal{E}_i + \mathcal{E}_j$ and $\mathcal{E}_i\mathcal{E}_j$ belong to $\mathcal{N}(\mathcal{S})$, which implies that $\mathcal{N}(\mathcal{S})$ can generate a linear subspace of $\mathcal{P}_n$. Thus, k operators $\mathcal{E}_1, \ldots, \mathcal{E}_k$ can be selected from $\mathcal{N}(\mathcal{S})$ such that the set

$$\Omega = \{\mathcal{G}_1, \ldots, \mathcal{G}_{N-k}, \mathcal{E}_1, \ldots, \mathcal{E}_k\} \tag{20}$$

is an independently commuting set. In terms of that described above, N operators $\mathcal{X}_1, \ldots, \mathcal{X}_N \in \mathcal{P}_N$ over $Cl(2\times 2)^{\otimes N}$ correspond to $2N$ elements $G^{(1)}, \ldots, G^{(2N)}$. For two operators $\mathcal{E}_i, \mathcal{G}_l \in \Omega$, there is an operator $\mathcal{X}_i$ satisfying $\mathcal{X}_i\mathcal{E}_i = -\mathcal{X}_i\mathcal{E}_i$ and $\mathcal{X}_i\mathcal{G}_l = \mathcal{G}_l\mathcal{X}_i$ for $1 \leqslant i \leqslant k$ and $1 \leqslant l \leqslant N-k$ respectively.

Since the encoding on N-qubits can be written as a tensor product of single qubit states [13], the encoder of QECC generates one of N-qubit logical states as follows:

$$\begin{aligned} |c\rangle_N &= \frac{1}{\sqrt{2^{N-k}}}\left(\prod_{l=1}^{N-k}(I+\mathcal{G}_l)\right)\mathcal{X}_1^{c_1}\cdots\mathcal{X}_k^{c_k}|\mathbf{0}_N\rangle) \\ &= \frac{1}{\sqrt{2^{N-k}}}\mathcal{X}_1^{c_1}\cdots\mathcal{X}_k^{c_k}\sum_{\mathcal{G}_l\in\mathcal{S}}\mathcal{G}_l|\mathbf{0}_N\rangle), \end{aligned} \tag{21}$$

where $|\mathbf{0}_N\rangle = |0\cdots 0\rangle$ is the initially prepared N-qubit state. $|c\rangle_N$ is a codeword obtained from k-qubit messages $|c\rangle_k = |c_1\cdots c_k\rangle$.

Above, we have provided a general approach for constructing quantum codes. In the following, we present examples for the QECC construction based on Hadamard matrices. A binary Hadamard matrix $H_N = (h_{ij})_{N\times N}$ is defined as a square matrix of the size $N \times N$. It satisfies two conditions, i.e. all entries are '1' or '-1' and two distinct rows are orthogonal. The Hadamard matrix has the following recursive relation:

$$H_N = H_2 \otimes H_{N/2} = \prod_{i=1}^{m}(I_{2^{m-i}} \otimes H_2 \otimes I_{2^{i-1}}), \tag{22}$$

where $N = 2^m, m \in \{1, 2, \ldots\}$, and H_2 is the 2×2 Hadamard matrix. Thus, one has

$$H_4 = H_2 \otimes H_2 = \begin{pmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{pmatrix}. \tag{23}$$

Generally, if 1 and -1 are replaced by 0 and 1, a Hadamard matrix is changed into a matrix over $\mathbb{Z}_2$. Taking a mapping $1 \to X$ and $-1 \to Z$, sequentially, a binary concatenated matrix may be gained with the correspondence in equation (2). According to H_4 in equation (23), with equation (3), one obtains a concatenated matrix $G_{4\times 8}$ that can be denoted by

$$G_{4\times 8} = \left(G_4^x G_4^z\right) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}, \tag{24}$$

where G_4^x and G_4^z are generated from H_4 with the elements mapping described. These maps are employed here and thereafter. By this means, one achieves a multilevel matrix $G_{N\times 2N}$ which is generated from the recursive relations of the Hadamard matrix H_N according to equation (22). It is easy to prove that all rows of $G_{4\times 8}$ satisfy the conditions in equation (10) and (11). In fact, for any two distinct rows $\vec{h}_i = (h_{i1}, \ldots, h_{i8})$ and $\vec{h}_j = (h_{j1}, \ldots, h_{j8})$ of $G_{4\times 8}$, one obtains $\sum_{k=1}^{4} h_{i,k}h_{i,4+k} = 0 \bmod 2$ and $\sum_{k=1}^{4}(h_{i,k}h_{j,4+k} + h_{j,k}h_{i,4+k}) = 0 \bmod 2$ with $i \neq j$. However, one cannot construct a quantum code using $G_{4\times 8}$ because of the quantum Hamming bound $\sum_{j=0}^{t}\binom{N}{j}3^j 2^k \leqslant 2^N$ [15]. Thus, the constructed matrices are not valid for all ratio k/N of quantum code under the quantum bound. We consider the case of $N > 5$ in the following.

Making use of the recursive relation of the Hadamard matrix in equation (22), one can easily get the concatenated matrix $G_{8\times 16}$ similar to $G_{4\times 8}$ in equation (24). Obviously, $\sum_{k=1}^{8} h_{i,k}h_{i,8+k} = 0 \bmod 2$ and $\sum_{k=1}^{8}(h_{i,k}h_{j,8+k} + h_{j,k}h_{i,8+k}) = 0 \bmod 2$ for any i, j with $i \neq j$. So $8-k$ rows of the generator matrix $\mathcal{G}_{8-k}$ of the stabilizer quantum code $[[8, k]]$ can be selected randomly from all rows of $G_{8\times 16}$. By similar means, a kind of generator matrices $\mathcal{G}_{N-k}$ used for the construction of the stabilizer quantum codes $[[N, k]]$ can be designed from the constructed Hadamard matrix H_N.

Corollary 2.1. *Making use of the recursive relation of the Hadamard matrix $H_{2^m} = H_2 \otimes H_{2^{m-1}}$, one may obtain the quantum codes with the parameters $[[N, k]]$, where $N = 2^m$ for $m \geqslant 3$.*

Proof . From the matrix H_{2^m}, one obtains the matrix $G_{2^m\times 2^{m+1}}$, whose rows can be denoted by $\vec{h}_i = (h_{i,1}, \ldots, h_{i,2^m}, h_{i,2^m+1}, \ldots, h_{i,2^{m+1}})$ for $1 \leqslant i \leqslant 2^m$. Because of the special structure of H_2, with theorem 2.2 it can be calculated that $\sum_{\tau=1}^{2^m} h_{i,\tau} \cdot h_{i,2^m+\tau} = 0 \bmod 2$ and $\sum_{\tau=1}^{2^m}(h_{i,\tau} \cdot h_{j,2^m+\tau} + h_{j,\tau} \cdot h_{i,2^m+\tau}) = 0 \bmod 2$, which implies that all rows of H_{2^m} satisfy the conditions in equations (10) and (11). Thus, any $2^m - k$ rows of $G_{2^m\times 2^{m+1}}$ can be composed of the generator matrix $\mathcal{G}_{2^m-k}$ of the stabilizer. With the $2N$ elements $\{G^{(1)}, \ldots, G^{(2N)}\}$ over the Clifford algebra $Cl(2N, \mathbb{C})$, quantum code $[[2^m, k]]$ for $m \geqslant 3$ may be obtained by making use of the encoder in equation (21). This completes the proof of the theorem. □

Corollary 2.2. *Since $H_{2^{2m+1}} = H_2 \otimes H_{4^m}$, all rows of the concatenated matrix $G_{2^{2m+1}\times 2^{2m+2}}$ of $H_{2^{2m+1}}$ satisfy the conditions in equation (10). Furthermore, $2^{2m+1} - k$ rows of the generator matrix $\mathcal{G}_{2^{2m+1}-k}$ of the stabilizer quantum code $[[2^{2m+1}, k]]$ over the Clifford algebra $Cl(2N, \mathbb{C})$ can be selected randomly from $2^{2m+1}-k$ rows of the concatenated matrix $G_{2^{2m+1}\times 2^{2m+2}}$ for $m \geqslant 2$.*

Corollary 2.3. *Since $H_{4^m} = H_4 \otimes H_{4^{m-1}}$, all rows of the concatenated matrix $G_{4^m\times 2\cdot 4^m}$ of H_{4^m} satisfy the conditions in equation (10). Consequently, $4^m - k$ rows of the generator matrix $\mathcal{G}_{4^m-k}$ of the stabilizer quantum code $[[4^m, k]]$ over the Clifford algebra $Cl(2N, \mathbb{C})$ can be selected from any $4^m - k$ rows of $G_{4^m\times 2\cdot 4^m}$ for $m \geqslant 2$.*

As examples, we present some stabilized quantum codes obtained using the proposed approaches. In terms of the obtained Hadamard matrix H_{4^m} for $m = 2$, one may obtain the quantum code $\mathcal{C}_{H_{16}} = \{[[16, k, d]] : k + d = 17, 1 \leqslant k \leqslant 15, 2 \leqslant d \leqslant 16\}$. Similarly, based on the obtained matrix $H_{2^{2m+1}}$ for $m = 2$, one obtains the quantum code

Table 1. Parameters of the constructed QECC and the corresponding matrices.

Parameters of QECC	Multilevel-constructed matrices
$[[2^m, k, d]]$	$H_{2^m} = H_2 \otimes H_{2^{m-1}}$
$[[4^m, k, d]]$	$H_{4^m} = H_4 \otimes H_{4^{m-1}}$
$[[2^{m+1}, k, d]]$	$H_{2^{m+1}} = H_2 \otimes H_{4^m}$
$[[4^m, k, d]]$	$H_{4^m} = H_4 \otimes H_{4^{m-1}}$

$\mathcal{C}_{H_{32}} = \{[[32, k, d]] : k + d = 33, 1 \leqslant k \leqslant 31, 2 \leqslant d \leqslant 32\}$. Furthermore, one can construct the quantum codes $[[4^m, k, d]]$ and $[[2^{2m+1}, k, d]]$ from H_{4^m} and $H_{2^{2m+1}}$ for $m \geqslant 3$, respectively, by making use of similar means. More stabilizer quantum codes based on the proposed approach will be presented in the near future. For clarity, we list some types of quantum codes with the parameters $\{[[N, k, d]] : k + d = N + 1\}$ constructed using the proposed approaches in table 1.

3. Conclusions

Based on the Clifford algebra, a new approach for constructing QECC is proposed. The key point of the proposed approach is to construct the generators of stabilizer of the quantum codes over the Clifford algebra. Two ways of creating generators of the quantum codes are presented. Since the recursive relations of the matrix representations of the Clifford algebra, the stabilizer quantum codes with arbitrary codeword length N can be constructed easily. In addition, some new codes, which are impossible based on the previous approaches, are obtained.

Acknowledgments

This work is supported by the Natural Science Foundation of China (grant numbers 60472018, 60773085) and is partly supported by the Ministry of Information and Communication under the IT Foreign Specialist Inviting Program supervised by IITA, the Korea Research Foundation KRF-2007-521-D00330, and Chonbuk National University in Korea.

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